Topology of Light's Darkness

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(Received 4 November 2008; published 6 April 2009)

We numerically study the topology of optical vortex lines (nodal lines) in volumes of optical speckle, modeled as superpositions of random plane waves. It is known that the vortex lines may be infinitely long, or form closed loops. Loops are occasionally threaded by infinite lines, or linked with other loops. We find the probability of a loop not being threaded decays exponentially with the length of the loop. This behavior has a similarity to scaling laws studied in superfluid turbulence, and polymers modeled as random walks.

DOI: 10.1103/PhysRevLett.102.143902 PACS numbers: 42.25.Hz, 02.40.Pc, 03.65.Vf, 42.30.Ms

As anyone who has tried to untangle a ball of string will agree, topological features are robust to transformations or distortions. As such, topological structures in physical media embody fundamental and robust properties. Here we study random, complex scalar fields which inherently contain tangles of nodal lines within them. Our physical example is three-dimensional optical speckle, formed when coherent light is scattered from a rough surface. Within any cross section there are points of perfect destructive interference, around which the phase increases or decreases by 2π. These points map out nodal lines in three dimensions, which are phase singularities [1,2] around which energy circulates (optical vortices). This physical situation is extremely general: tangled quantized lines occur in other three-dimensional scalar fields, such as those describing superfluid turbulence [3–5], including Bose-Einstein condensates [6], and cosmology [7]. Similar tangles of nodal lines are also found in vector fields such as randomly polarized optical fields [8,9].

In laser speckle [10] much is known about optical vortices [11–13]. Recently, we found that, in random wave superpositions, vortex lines have a large-scale self-similarity characteristic of a Brownian random walk [14]. For a Gaussian angular power spectrum, approximately 73% of the vortex line length is accounted for by infinite lines that percolate through the entire volume, and the remaining 27% of the line length in loops. For loops with perimeters greater than the coherence length of the field, the loop length distribution is described by a power law consistent with the self-similarity of the tangle. However, previous work has not addressed the topology of these random vortex lines or whether they naturally form links and knots, similar to those generated by specific superpositions of laser modes [15,16].

In this work we model random speckle fields over large volumes and study the topological features of the vortex tangle. In previous numerical studies of vortices in superfluid turbulence, topological properties have been found to obey scaling laws, such as the distribution of lengths of closed vortex loops [17], and distributions of crossing and linking numbers from projection [4]. In this optical counterpart to that work, we consider both the length distribution of closed vortex loops in 3D optical speckle, which we

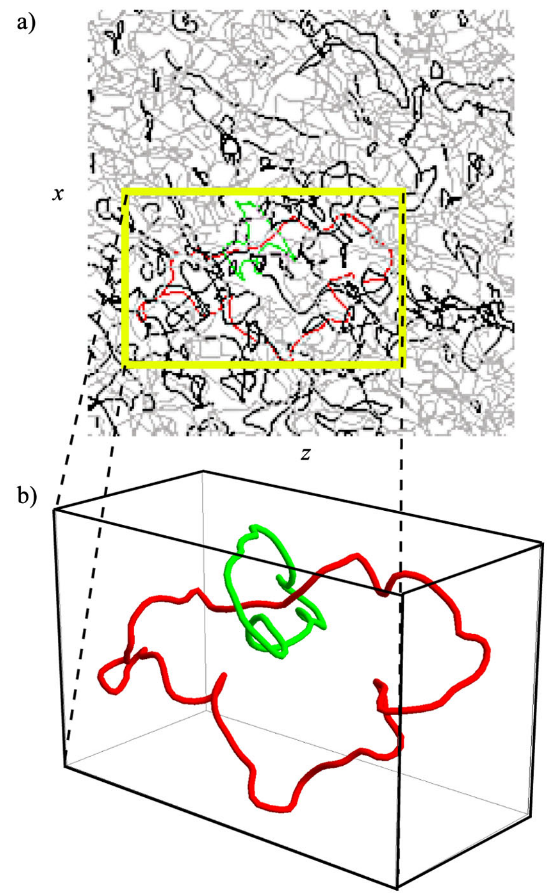

FIG. 1 (color online). Topology in the tangle. (a) shows a projection of vortex lines in a cubic volume of side length 3Λ. [Λ is coherence length of the field distribution defined by Eq. (1)]. Sections of infinite lines are shown in gray, loops in black and the two highlighted (linked) loops in red and green (b) shows a view of the vortex link.

find to have a Brownian fractal scaling, and the distribution of vortex loops linked or threaded by other vortices. We find this linking and threading to be exponentially distributed with loop length, reminiscent of knotting probabilities observed in random walk models of polymer loops [18]. Furthermore, our method for establishing whether loops are linked, threaded, or knotted apply to any kind of line tangle. An example of a vortex link within a simulated optical vortex tangle is shown in Fig. 1. Since the vortex handedness is with reference to the z direction, positions where the vortex line has a local maximum or minimum in the z coordinate are sometimes referred to as points of vortex pair creation and annihilation. However, this temporal language is strictly not appropriate to static vortex lines.

In any simulated system, the finite size of the computer memory limits the volume of the simulation, and thus any vortex line that penetrates the volume has an ill-defined topology. We overcome this limitation by setting plane waves to lie on a square grid in transverse k-space [19]. This leads to an interference pattern that is both transversely and axially periodic in real space by the Talbot effect. These Talbot cells [14] tile all space, giving a speckle field throughout which the angular power spectrum is the same. Consequently, when a vortex line approaches the edge of the Talbot cell, its path can be followed by periodicity. Such a line may eventually return to the original cell to form a closed loop, or return to the corresponding starting point in a different cell, corresponding to an infinite periodic line.

For each plane wave component in the k-space grid we assign normally distributed random values to the real and imaginary amplitudes. This is modulated by a multiplicative Gaussian envelope of standard deviation K_σ giving a Gaussian power spectrum, representative of a propagating optical field above a scattering surface. The spacing of the k-space grid, δk, sets the lateral period x_T, and axial period z_T, as $2\pi/\delta k$ and $4\pi k_0/\delta k^2$ respectively, where k_0 is the free space wave number. Although the aspect ratio of the resulting interference pattern scales with δk, provided that the situation remains paraxial, the topological scaling rules we describe below remain the same. Within our model we set the ratio of K_σ/k_0 to be 1/330. It is then natural to define the coherence length Λ, for the resulting speckle pattern which scales with the numerical aperture of the system as

$$\Lambda = \sqrt[3]{\Lambda_{x,y}\Lambda_z} \qquad (1)$$

where $\Lambda_{x,y} = \lambda(k_0/K_\sigma)$ and $\Lambda_z = \lambda(k_0/K_\sigma)^2$. Thus $x_T = \Lambda_{x,y}(K_\sigma/\delta k)$ and $z_T = 2\Lambda_z(K_\sigma/\delta k)^2$. The resulting Talbot cell is representative of a volume of the speckle field through which the angular spectrum is unchanged.

The topological properties of the random Talbot cell approach those of an infinite, nonperiodic interference pattern when $K_\sigma/\delta k$ becomes large, which is equivalent to increasing the number of participating plane waves [approximately given by $\pi(K_\sigma/\delta k)^2$]. We vary the size of the k-space grid from 23×23 to 41×41 and calculate the resulting interference pattern over a Talbot cell of $800 \times 800 \times 10\,000$ voxels. The vortex tangle in Fig. 1(a) corresponds to a small section of one Talbot cell of size $x_T/2$ by $z_T/24$.

Despite their robustness, nodal lines are challenging to locate numerically and identify topologically. This is especially true when a curve is only known at discrete positions. The plane projection of any tangle of curves has many crossings; see Fig. 1(a). As described below, the topological linking between any pair of curves (or knotting of a single curve) is determined by the sequence of signed crossings [20] along the curve (the sign is determined by the right-hand rule) [20].

Characterization of the topology requires locating the crossing points of two vortex line curves $\mathbf{a}$ and $\mathbf{b}$, with respect to projection in some plane (such as the xy plane). Between every pair of points on the curves, $\mathbf{a}(i)$ and $\mathbf{b}(j)$, there is a chord, which has a direction $\theta_{\mathbf{a},\mathbf{b}}(i,j) \equiv \arg[\{\mathbf{b}(j) - \mathbf{a}(i)\} \cdot (1, i, 0)]$. Figure 2 shows one example of a link and the corresponding chord plot. The singularities in the chord angle plot mark the crossing points between the two loops. Their presence is understood by fixing one end of a chord at a crossing point in the projection of $\mathbf{a}$, and moving the other end of the chord along $\mathbf{b}(j)$. As the point on $\mathbf{b}$ approaches the crossing point the chord shrinks in length, and as it passes the crossing point the chord flips by π. At the crossing point the chord length is zero and its angle is undefined.

The sense of increase of $\theta_{\mathbf{a},\mathbf{b}}$ defines the sign of the chord singularity, $s = \pm 1$. The sign of the crossing is sp, where $p = \mathrm{sgn}[\{\mathbf{b}(j) - \mathbf{a}(i)\} \cdot (0, 0, 1)]$, i.e., the sense of the chord normal to the projection plane. The topology of the curve pair is completely defined by the configuration of signed crossings; for instance, the linking number l between the two curves is half the sum of signed crossings.

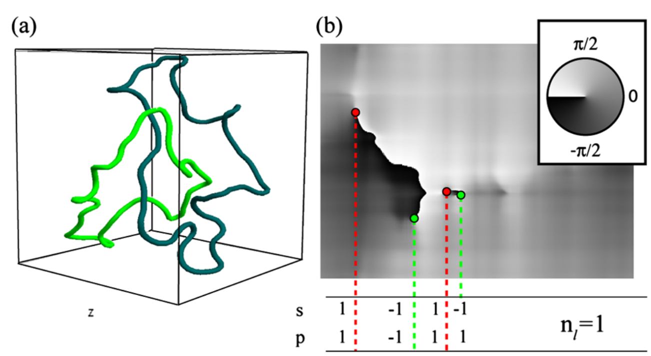

FIG. 2 (color online). Linked vortex loops. (a) Shows a linked pair of vortex loops, $\mathbf{a}$, $\mathbf{b}$ in a simulated speckle field. (b) Shows the chord angle $\theta_{\mathbf{a},\mathbf{b}}(i,j)$ pairs of points on the loops projected into viewing plane. The singularities in the chord angle correspond to signed crossings with values s and p given in the table.

The search for knotted vortex lines is implemented in a similar way, but with chords between points on the same curve. Again projecting into a plane, the projected chord angle between points i and j on curve $\mathbf{a}$ is defined $\theta_{\mathbf{a},\mathbf{a}}(i, j) \equiv \arg[\{\mathbf{a}(i) - \mathbf{a}(j)\} \cdot (1, i, 0)]$. There is a diagonal antisymmetry $\theta_{\mathbf{a},\mathbf{a}}(j, i) = \theta_{\mathbf{a},\mathbf{a}}(i, j) + \pi$, and, in particular, there is a discontinuity of π along the line $i = j$.

Knots are harder to identify than links. We calculate the Alexander polynomial for each curve [20], using the method described in Ref. [18]. The Alexander polynomial is a Laurent polynomial in one variable with integer coefficients, easily calculated from the crossing data of a curve, which distinguishes most knots from each other (although some share the same polynomial).

A natural measure of the size of a closed loop is its radius of gyration R_g, defined as the mean distance of all points on the loop from its center of mass. For a circular planar loop of length L, $R_g = L/2\pi$. For more complicated nonplanar loops, R_g grows more slowly with L. In general, the projection of larger loops appears similar in every direction, and we find that R_g scales approximately as $L^{1/2}$, as indicated in Fig. 3, indicative of Brownian fractality. A pair of isotropic loops whose gyration spheres intersect are strong candidates for a link, and the range of loop sizes over which we have observed links is comparable to the range of Brownian scaling.

The total data we have simulated correspond to a speckle field volume of approximately $1.5 \times 10^5 \Lambda^3$. This data took roughly 3000 h of CPU time on a cluster of P4 processors to generate. Within this total volume we have identified unthreaded loops, loops threaded by periodic lines and loops linked with other loops. The fraction of vortex line length in a Talbot cell in closed loops, and the number density of loops threaded by infinite lines, appears to be stable with respect to $K_\sigma/\delta k$. This stability suggests the periodic boundary conditions do not strongly affect the statistics of the topological features. We also note that our largest k-space grids (41×41) correspond to periodicities of $x_T = 6\Lambda_{x,y}$ and $z_T = 72\Lambda_z$, indicating that most of our loops are contained within a single Talbot cell, again suggesting that the periodicity is unlikely to be a significant factor in the statistics of threaded and linked loops.

Figure 4 shows the likelihood that a vortex loop is threaded by another vortex line and when that vortex line is part of another loop (i.e., a link). Unsurprisingly, larger loops are more likely to be threaded and linked. Indeed the smallest threaded loop we have found is $L_0 \approx 2\Lambda$ in length. Above this minimum size, we find that the probability P_{unthread} of *not* being threaded decays exponentially with the length, L, of the loop,

$$P_{\mathrm{unthread}} \sim \exp(-L/T\Lambda), \qquad (2)$$

where $T \approx 30$ for threading any by vortex line and $T \approx 185$ for threading by another loop. This apparent exponential decay of topological isolation is similar to that found in the literature for other systems of random tangles, such as polymers [18,21]. It is numerically observed, across a range of models of random polymer loops, that the probability that a loop of length L is unknotted is $\exp(-L/AP)$ where P is the persistence length of the polymer and A is a dimensionless number that depends on the details of the specific model. A ranges from 300 for random piecewise linear loops with segments of equal length to 10 000 for closed random walks on cubic lattices [18].

The persistence length of a curve is defined as the exponential decay distance for the correlation in the direction of the tangents along the curve. For our vortex lines, we find this to be $P = 0.17\Lambda$, comparable to the characteristic length scale found in Ref. [14] and the corresponding radius at which loops become planar in Fig. 3.

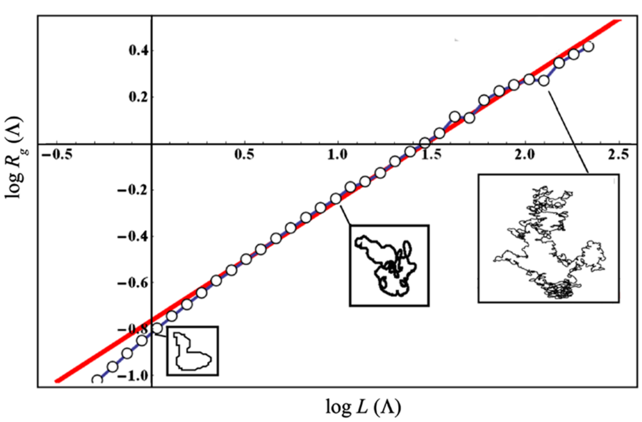

FIG. 3 (color online). A Log-log plot of the radius of gyration R_g against loop length, L. For illustrative purposes, three loops of different lengths are shown. The red line is a fit to the straight line section (0.2–2.0) on the log L scale and has a gradient of 0.52. Links between two loops were only found for $\log(L) > 0.2$.

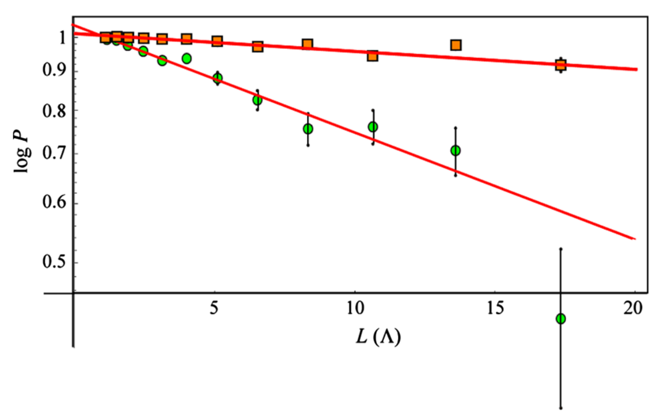

FIG. 4 (color online). A plot of the probability P of loops of a given length L, not to be threaded by vortex lines of any type (green circles) or linked to another loop (orange squares). The red lines show fit to straight lines indicating an exponential decay lengths of approximately 30Λ and 185Λ for not being threaded by a vortex line and vortex loop, respectively.

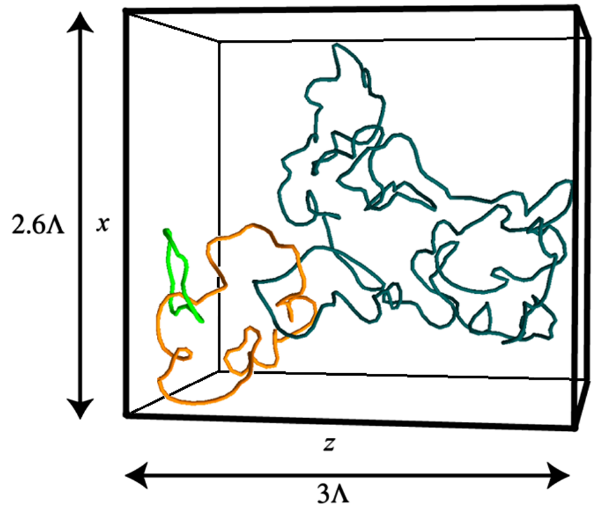

FIG. 5 (color online). A three component link found within our simulated speckle field. Within a total calculated volume of approximately $1.5 \times 10^5 \Lambda^3$ only 10 such features were found.

Of more complex links, we have found a small number (about 10) of a single larger loop being linked to several smaller ones; see Fig. 5. We have also found a single example of a link where one loop threads the other twice. However, our survey of overlapping loops does not search for links whose net linking number is zero, such as the Whitehead link or Borromean rings [20]. Somewhat to our surprise, a search of our data (approximately 500 000 loops) did not reveal any knots. This may be due to their extreme rarity, memory restrictions in our sampling, or less likely, a hidden prohibition of knots in our ensemble of random fields. It is tempting to conjecture that the probability distribution of a vortex loop *not* being knotted, P_{unknot} should, as with the links and polymer knots, have an exponential dependence

$$P_{\text{unknot}} \sim \exp(-L/K\Lambda). \quad (3)$$

It follows that the probability for none of a large number N of loops to be knotted is approximately $\exp(-\sum_{i=1}^{N} L_i/K\Lambda)$. Considering only the loops in the size range over which we have found links ($L > 2\Lambda$), and restricting the search to the data sets with highest number of plane wave components, results in approximately 15 000 candidate loops, giving a total loop length of about $1.2 \times 10^5 \Lambda$. This suggests a lower bound for K of 1.2×10^5. From the loop size density given in Ref. [14] we anticipate that a loop of this length occurs once in a volume of $10^{12}\Lambda^3$—some 10^7 times larger than our own search volume.

Topological scaling is an important property of large tangles of lines in a range of physical systems. We have established numerically here that in random optical speckle, modeled by paraxial linear superpositions of plane waves, nodal vortex loops obey scaling laws. Their length distribution is Brownian, and the probability that a loop is *not* threaded by another vortex line decays exponentially with loop length. Our search of computer-generated data has not found any knots, possibly due to the fact that the simulated Talbot cell is smaller than the enclosing volume of the smallest random vortex knots. However, the topological scaling of vortex loop linking and threading presents a continuum, linear wave comparison to scalings of tangles in polymer tangles and superfluid turbulence.

We thank Robert King and Professor Shura Grosberg for helpful discussions. This work was supported by the Leverhulme Trust and the Royal Society of London.

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