Polarization Singularities in 2D and 3D Speckle Fields

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(Received 22 January 2008; revised manuscript received 9 April 2008; published 22 May 2008)

The 3D structure of randomly polarized light fields is exemplified by its polarization singularities: lines along which the polarization is purely circular (C lines) and surfaces on which the polarization is linear (L surfaces). We visualize these polarization singularities experimentally in vector laser speckle fields, and in numerical simulations of random wave superpositions. Our results confirm previous analytical predictions [M. R. Dennis, Opt. Commun. 213, 201 (2002)] regarding the statistical distribution of types of C points and relate their 2D properties to their 3D structure.

DOI: 10.1103/PhysRevLett.100.203902 PACS numbers: 42.25.Ja, 42.30.Ms

Topological defects play an important role in many physical phenomena [1], occurring in ordered media [2] and in propagating waves such as quantum wave functions [3] and, as we study here, light [4]. In scalar fields representing uniformly polarized optical beams, the defects are the widely studied optical phase singularities [4–6], also called nodes and optical vortices. Optical vortex lines in random 3D wave fields (speckle fields) were recently found to have Brownian scaling properties [7]. However, the most general optical beams also have position-dependent polarization, described by a complex vector field. In this case the singularities are places in the cross section of pure circular or linear polarization. The resulting fields have many subtleties not present in the scalar case.

Within any transverse plane of a paraxial optical field it is useful to characterize the smoothly varying polarization by streamlines oriented along the major axis of the polarization ellipse. Around every C point these streamlines rotate by ±π, illustrated in Fig. 1 [8]. The positive index singularities occur in two forms: the lemon type, on which only one streamline terminates, and the less common monstar, on which three straight streamlines terminate. On negative-index singularities, termed stars, there are always three streamlines that terminate [9]. These singularities occur throughout polarization optics, e.g., tightly focused beams and near-field optics [10], skylight [11], and crystal optics [12,13]. As these fields propagate, the polarization changes continuously and the C points sweep out C lines.

In paraxial optical fields with random polarization (vector speckle fields), the field is dominated by its transverse components and the volume is filled with a complicated network of C lines, and surfaces of linear polarization (L surfaces) separating those of opposite handedness. These C lines are similar in their characteristics to the optical vortex lines encountered in the speckle patterns of scalar fields [7,14]. Here, we establish the statistical frequency of stars, monstars, and lemons, and compare our results to previous analytical predictions [9,15].

In his seminal work [4,8], Nye recognized that throughout the volume the C lines follow curved paths, exhibiting turning points and sometimes closed loops. At turning points the singularity’s direction reverses with respect to the propagation direction, and its index changes sign. The singularity type therefore switches from star to lemon and vice versa, but near this transition the lemon singularity becomes a monstar. Away from these turning points, a lemon can switch to a monstar and back again.

The polarization state at each point of a light beam is completely described by the Stokes parameters (S0, S1, S2, S3) [16]. Operationally these parameters can be measured from the intensity of the light associated with different polarization states: S0 is merely the overall intensity, and the others are given by

\[ S_1 = I_{90^\circ} - I_{90^\circ}, \quad S_2 = I_{45^\circ} - I_{135^\circ}, \quad S_3 = I_{\text{left}} - I_{\text{right}}. \]

(1)

where \( I_\theta \) is the intensity of the linear polarization at angle \( \theta \), and \( I_{\text{left}} \) (right) the intensity of the left (right) circular polarization. L surfaces are those on which \( S_3 = 0 \) and C lines are defined as the intercept of the loci \( S_1 = 0 \) and \( S_2 = 0 \). As described later, the type of C line singularity may be determined using the Stokes parameters.

FIG. 1 (color online). In random vector fields C points are classified by the streamlines of their immediate surroundings into stars (S), monstars (M, short for lemon-stars), and lemons (L). The polarizations may be either right handed or left handed, but C points of opposite handedness are always separated by an L line.
In our experiment, illustrated in Fig. 2, the vector fields are generated by the interference of two orthogonally polarized speckle patterns created by illuminating two neighboring regions of a ground glass screen with expanded beams derived from the same HeNe laser. The orthogonally polarized fields are superimposed using a polarizing beam splitter. A 12-bit CCD camera and imaging lens, mounted on a motorized stage, are then interpolated to quadratic order and subjected to a zero-finding algorithm to locate the C point precisely. The singularity index of the C point is determined by the sign of $D_I$,

$$D_I = S_{1,x}S_{2,y} - S_{1,y}S_{2,x},$$

where $x,y$ subscripts denote spatial derivatives. If $D_I < 0$ then the singularity is a star, if $D_I > 0$ the singularity is a monstar or lemon. Monstars are then distinguished from lemons by the sign of $D_L$ [15]:

$$D_L = [(2S_{1,y} + S_{2,x})^2 - 3S_{2,y}(2S_{1,x} - S_{2,y})] \times [(2S_{1,x} - S_{2,y})^2 + 3S_{2,x}(2S_{1,y} + S_{2,x})] - (2S_{1,x}S_{1,y} + S_{1,y}S_{2,x} - S_{1,y}S_{2,y} + 4S_{2,x}S_{2,y})^2.$$  

If three straight streamlines meet at the singularity then $D_L > 0$ and the singularity is a star or monstar, if $D_L < 0$ then the singularity is a lemon. Figure 3 shows the experimentally measured polarization streamlines in the vicinity of a number of C points.

A third classification, the contour classification, divides $C$ points into hyperbolic or elliptic types [4]. This classification is related to the shape of contours of the polarization ellipse axis lengths, which form a double cone structure near the singularity [4,9,20], and is an additional quantity which our observations determine. A C point is elliptic or hyperbolic depending on the sign of $D_C$, given by [15]

$$D_C = (S_{1,x}S_{2,y} - S_{1,y}S_{2,x})^2 - (S_{1,x}S_{0,y} - S_{1,y}S_{0,x})^2$$

$$- (S_{0,x}S_{2,y} - S_{0,y}S_{2,x})^2,$$

where $D_C < (>)0$ for hyperbolic (elliptic) $C$ points.
The density of \( C \) points and the ratios of their different types in an isotropic random wave model were calculated in Ref. [15], based on earlier work by Berry and Hannay on umbilic points on random surfaces [9]. The density of \( C \) points per unit coherence area was calculated to be \( \frac{4}{0.0003} \) per unit coherence area and the star:lemon:monstar ratio is 50:44:72:5. Since \( C \) points in random vector speckle fields are the vortices in the independent, identically distributed left and right circular components, the \( C \) point density is twice the underlying vortex density of \( \frac{2}{0.0255} \), which has been verified experimentally [21].

Polarization singularity densities have previously been investigated in experiments [22], but with data sets that were too small to give average densities. Table I shows the breakdown of observed and simulated \( C \) points into their singularity type, in the index or line classification (lemon, monstar, or star) and contour classification (elliptic or hyperbolic). Our quoted errors are a combination of statistical uncertainty based on the finite numbers of singularities found and by varying the radius of the line integral around the \( C \) points. However, in all cases our agreement with the analytic statistics predictions [15] is excellent.

Additionally, we find the density of numerically simulated \( C \) points is \( 12.51 \Lambda_T^{-2} \), which is close to the predicted value of \( 4\pi \Lambda_T^{-2} = 12.57 \Lambda_T^{-2} \).

In our previous numerical study of optical vortices in scalar speckle [7], we found that about 73% of the vortex length is in infinite lines, the remainder in closed loops. For length scales above the coherence length of the field, the vortex lines scale as fractals of dimension 2. Since random polarization fields (theoretically and experimentally) are superpositions of independent random fields in their left and right circular components—in which the \( C \) lines are vortices of the appropriate component—we conjecture that \( C \) lines in random vector speckle similarly scale like Brownian random walks. However, the \( C \) point type, being

<table>
<thead>
<tr>
<th>Singularity type</th>
<th>Simulation</th>
<th>Experiment</th>
<th>Dennis [15]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Star</td>
<td>0.501 ± 0.002</td>
<td>0.506 ± 0.003</td>
<td>0.500</td>
</tr>
<tr>
<td>Lemon</td>
<td>0.450 ± 0.003</td>
<td>0.443 ± 0.002</td>
<td>0.447</td>
</tr>
<tr>
<td>Monstar</td>
<td>0.049 ± 0.002</td>
<td>0.050 ± 0.003</td>
<td>0.053</td>
</tr>
<tr>
<td>Star ( E/H )</td>
<td>1.035 ± 0.054</td>
<td>1.073 ± 0.078</td>
<td>1.000</td>
</tr>
<tr>
<td>Lemon ( E/H )</td>
<td>1.133 ± 0.078</td>
<td>1.086 ± 0.112</td>
<td>1.104</td>
</tr>
<tr>
<td>Monstar ( E/H )</td>
<td>0.418 ± 0.056</td>
<td>0.487 ± 0.031</td>
<td>0.404</td>
</tr>
</tbody>
</table>

FIG. 3 (color online). Examples of four \( C \) points and nearby polarization streamlines as measured in our experiment. Part I is lemon type, part II is star, part III is monstar, and part IV is a cross section near a star-monstar transformation.
TABLE II. Fraction of C lines of different type and ratio of
elliptic to hyperbolic (E/H) types, evaluated over numerically
simulated (5 × λ3) and experimentally observed (2 × 1.25λ3)
volumes.

<table>
<thead>
<tr>
<th>Singularity type</th>
<th>Simulation</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Star</td>
<td>0.504 ± 0.008</td>
<td>0.496 ± 0.011</td>
</tr>
<tr>
<td>Lemon</td>
<td>0.420 ± 0.005</td>
<td>0.422 ± 0.010</td>
</tr>
<tr>
<td>Monstar</td>
<td>0.076 ± 0.005</td>
<td>0.082 ± 0.011</td>
</tr>
<tr>
<td>Star E/H</td>
<td>0.814 ± 0.039</td>
<td>0.727 ± 0.081</td>
</tr>
<tr>
<td>Lemon E/H</td>
<td>0.944 ± 0.058</td>
<td>0.787 ± 0.010</td>
</tr>
<tr>
<td>Monstar E/H</td>
<td>0.260 ± 0.028</td>
<td>0.274 ± 0.038</td>
</tr>
</tbody>
</table>

a vector property of the singularity, cannot be deduced
from the underlying scalar field.

Figure 4 shows a typical experimental observation of the
C line [4(a)] and the associated L surface [4(b)] structure
within a random vector speckle field. They are plotted over
a natural coherence volume \( \lambda^3 = \lambda^2 \lambda_z \), where \( \lambda_z \) is the
longitudinal coherence length. The C lines are color coded
to denote stars, monstars, and lemons of both right- and
left-handed circular polarization. As anticipated [4,8], we
note that points on C line loops are mainly stars and
lemons, with short monstar sections at the maximum and
minimum z extent. Monstars also frequently occur within
the lemon section of the C line, far away from the turning
points, and it is these monstars that appear most common.
Figure 4(c) shows a small section of the same volume
highlighting one C line loop and the linearly polarized L
surface in its vicinity, separating it from the C lines of
opposite handedness. Sections I–IV are those for which the
streamlines are plotted in Fig. 3.

By rescaling z, so the coherence lengths \( \lambda_T = \lambda_z = \lambda \),
we expect the tangent direction of C lines to be uniformly
distributed. From the solution to the classical “Buffon
needle problem” [23], generalized to lines in 3D [24],
the C line density per unit coherence volume ought to be
twice the C point density in transverse section, i.e.,
\( 8 \pi \lambda^{-2} = 25.133 \lambda^{-2} \). Our numerical simulations give
this line density as 25.76\( \lambda^{-2} \).

We also consider the distribution of singularity type as a
fraction of C line length, given in Table II. These are
different from the results of Table I, where the density is
weighted in proportion to the z component of the C line
tangent. In particular, since monstars occur when C lines
are approximately perpendicular to the propagation
direction, their 3D weighting is higher (7.6\%) than in transverse
sections (5.3\%). It appears also that the elliptic weighting
dominates over the hyperbolic. As yet there are no analytic
calculations for these numbers to match the results of
Table II, although for fields where the polarization ellipse
plane orientation is also random [25], we expect the 3D
weightings to match Table I.

In conclusion, we have experimentally visualized polariza-
tion singularities—C lines and L surfaces—in random
vector speckle fields. We reasoned that the large scale
structure is identical to that recently reported for vortices
in scalar speckle fields. In addition to verifying the singu-
larit y type distribution in 2D against theoretical predic-
tions, we have new experimental and simulated predictions
for their three-dimensional counterparts, including through
the Buffon needle problem a link between their 2D and 3D
statistics.

This work was supported by the Leverhume Trust and
F.F. was supported by the Deutsche Forschungsgemein-
schaft (DFG). M. R. D. is supported by the Royal Society.